1. **Recovering a Curve from its Curvature**
   Recall that a planar curve is uniquely determined by its curvature as a function of arc length and starting position and starting direction.
   For parts 1b-1e, use Maple or some software to make the plots. You can also do Maple for 1a to help integrate if you like.

   (a) Find some 2-D curve $\mathbf{r}$ on $[1, \infty)$ such that its signed curvature with respect to arc length is given by $\kappa_s(s) = 1/s$ for $s > 0$. and $\mathbf{r}(1) = [0, 0]$ and $\mathbf{r}'(1) = [1, 0]$.

   (b) Plot the curve $\mathbf{r}$ for $t \in [1, 10]$.

   (c) Plot the curve $\mathbf{r}$ for $t \in [1, 100]$.

   (d) Plot the curve $\mathbf{r}$ for $t \in [1, 1000]$.

   (e) Plot the curve $\mathbf{r}$ for $t \in [1, 10000]$.

   (f) Explain how the curvature is determining the curve

2. **Some Handy Facts** Show the following. These will be useful for later problems.

   (a) Show that for a 2-D vector $\mathbf{a}$ that $\left((\mathbf{a})^{\perp}\right)^{\perp} = -\mathbf{a}$

   (b) Show that if $\mathbf{r}$ is a planar curve parameterized by arc length that $\mathbf{r}''(t) = \kappa_s(t)\,(\mathbf{r}'(t))^{\perp}$

3. **Normal Offsets.**
   Suppose $\mathbf{r}(t)$ is a regular smooth curve with signed curvature $\kappa_s(t)$. Let α be a scalar and define

$$\mathbf{q}(t) = \mathbf{r}(t) + \alpha \frac{1}{||\mathbf{r}'(t)||} (\mathbf{r}(t))^{\perp}$$

   The curve $\mathbf{q}$ is called a normal offset of $\mathbf{r}$.

   (a) Plot some normal offsets of the curve $\mathbf{r}(t) = (t, t^2)$ for $t \in [-1, 1]$. Make plots for the following alphas: $\alpha = -2, -1, 0, 1/4, 1/2, 1$; and more if you like. (Use Maple or some software.)

   (b) Supposing that $\mathbf{r}$ is parameterized by arc length, find a simplified formula for $\mathbf{q}'(t)$ in terms of $\mathbf{r}$, $\mathbf{r}'$, κ_s and α. Show that $\mathbf{q}'(t)$ is parallel to $\mathbf{r}'(t)$.

   (c) Using the formula you found in the previous part and assuming $\kappa_s > 0$, find conditions on α so that $\mathbf{q}$ is regular.

   (d) Explain what happens in the pictures you made in part (a) when $\mathbf{q}$ is not regular.

4. **Involutes and Evolutes:**

   - The evolute of a regular smooth curve $\mathbf{q}(t)$ on $[a, b]$ is given by

$$\mathbf{e}(t) = \mathbf{q}(t) + R_s(t) \frac{1}{||\mathbf{q}'(t)||} (\mathbf{q}'(t))^{\perp}$$

     where $R_s(t) = 1/\kappa_s(t)$ is the signed radius of curvature of $\mathbf{q}$.

- The *general* involute of a regular smooth curve $\mathbf{r}(t)$ on $[a, b]$ is given by

$$\mathbf{v}(t) = \mathbf{r}(t) - (s(t) + l)\frac{1}{||\mathbf{r}'(t)||}\mathbf{r}'(t)$$

  where $s(t)$ is the arc length of $\mathbf{r}$ from a to t and l is some constant corresponding to the initial string length.

(a) Involutes of Evolutes: Let $\mathbf{q}$ be parameterized by arc length and let $\mathbf{e}$ be its evolute.

  i. Show that $\mathbf{e}'(t)$ is perpendicular to $\mathbf{q}'(t)$

  ii. Show that the involute of $\mathbf{e}(t)$ is equal to $\mathbf{q}(t)$ for the appropriate choice of the constant l.

(b) Evolutes of Involutes: Let $\mathbf{r}$ be parameterized by arc length and let $\mathbf{v}$ be its involute.

  i. Show that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{v}'(t)$

  ii. Show that the evolute of $\mathbf{v}(t)$ is equal to $\mathbf{r}(t)$

(c) Make a sketch that illustrates how involutes and evolutes are inverse processes. Label all the parts.

5. **Torsion/Curvature = constant**

(a) Suppose we want a space curve with constant curvature κ and constant torsion τ so that $\tau/\kappa = L$ for some constant L. Find constants a and b so that the helix $\mathbf{r}(t) = [a\cos(t), a\cos(t), bt]$ has such curvatures and torsions.

(b) For the space curve, $\mathbf{r}(t) = (\cos(t) + t\sin(t), \sin(t) - t\cos(t), t^2/2)$, calculate the curvature and torsion and show that the ratio, *torsion/curvature*, is constant. (Notice that the projection into the $x - y$ plane of this curve is is an involute of a circle.)